Corollary 3. It is clear from the proof that, when k is odd, the formal solution contains no logarithms.

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AN ASYMPTOTIC ANALYSIS OF THE FORCED OSCILLATIONS IN SYSTEMS WITH SLOWLY VARYING PARAMETERS*

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The oscillations in weakly non-linear systems with slowly varying parameters are investigated. For periodically varying parameters, a spectral analysis is made of the steady-state oscillations in order to obtain reasonably simple analytical results. Special attention is paid to the cases when some natural frequencies vary over a much wider range than the frequency of parameter variation.

The usual basic methods for analysing such problems /1-3/ are not suitable of the present purpose, especially when the parameters vary over a wide range. A rather different scheme for analysing the system of differential equations is proposed below. The matrizant (Green's function) of the linear problem is written in a form which ensures faster convergence than in the WKB method and of the procedure for the asymptotic evaluation of the required quantities /1, 2/. Even to a first approximation, the results differ from those of /1, 2/, and differ the more, the greater the range of variation of the parameters. The non-linear forces are taken into account by successive approximation

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with partial linearization at each step. The slowly varying coefficients of the linearized part correct the parameters of the linear operator and are functionals of the relevant approximate solution. In some cases, this functional problem can be reduced to an ordinary equation in several unknowns. A one-dimensional oscillatory system is studied in more detail in this context, with periodically varying rigidity and cubic non-linearity, under the action of a single-frequency force excitation.

1. The linear approximation. Consider the equation

$$L_{kn}x_n(t) = F_k(t),\ x_k(0) = x_k^{\cdot}(0) = 0 \tag{1.1}$$
$$L_{kn} = \delta_{kn}\frac{d}{dt}m_k\frac{d}{dt} + \sqrt{m_k m_n}\left(2R_{kn}\frac{d}{dt} + U_{kn}\right)$$
$$0 \leqslant t \leqslant T,\ k, n = 1, 2, \ldots, N,\ \delta_{kn} = \begin{cases}1, & k = n\\ 0, & k \neq n\end{cases}$$

Here and throughout, we understand summation over repeated Latin subscripts, running from 1 to N. The matrices R and U are positive definite and symmetric (though this is not essential for the essence of the method), while the functions $R_{nk}(t)$, $U_{nk}(t)$, $m_k(t) \geqslant m_0 > 0$, which will often be denoted by $C(t)$, are differentiable a sufficient number of times (i.e., all the derivatives used below are bounded). The functions $F_k(t)$ are bounded. To reduce the number of subscripts, we put $F_k = \delta_{k1}F_1$ and we shall seek the inverse operators L_{k1}^{-1}.

Let $\lambda_\alpha(t)$, $\Psi_k^\alpha(t)$ $(\alpha = 1, 2, \ldots, N)$ be the eigenvalues and eigenvectors of the matrix U, where $\lambda_\alpha \geqslant \lambda_0 > 0$, $|\lambda_\alpha - \lambda_\beta| \geqslant \lambda_0$ for all $\alpha \neq \beta$. We define the vectors Ψ_k^α by means of the cofactors of $(U - \lambda_\alpha)$:

$$P_k^\alpha = \frac{\partial U(\lambda)}{\partial U_{k1}}\bigg|_{\lambda=\lambda_\alpha},\quad P_{kn}^\alpha = \frac{\partial^2 U(\lambda)}{\partial U_{kn}\partial U_{11}}\bigg|_{\lambda=\lambda_\alpha},\quad U(\lambda) = \det(U - \lambda)$$

We shall assume that $|P_1^\alpha(t)| \geqslant P_0 > 0$. We put

$$\Psi_k^\alpha = P_k^\alpha\Psi_\alpha/P_1^\alpha,\ \Psi_\alpha = \left[P_1^\alpha\Big/\Big(\prod_{\beta\neq\alpha}(\lambda_\beta - \lambda_\alpha)\Big)\right]^{1/2}$$

We will seek the solution of Eq.(1.1) in the form

$$x_k = L_{k1}^{-1}F_1 = \sum_{1\leqslant\alpha\leqslant N}(m_k\omega_\alpha)^{-1/2}\left\{\int_0^t dt'\left[A_k^\alpha(t)\sin\left(\int_{t'}^t \omega_\alpha\,dt\right) + B_k^\alpha(t)\cos\left(\int_{t'}^t\omega_\alpha\,dt\right)\right]\exp\left(-\int_{t'}^t\gamma_\alpha\,dt\right)\frac{W_\alpha(t')F_1(t')}{[m_1(t')\omega_\alpha(t')]^{1/2}}\right\} \tag{1.2}$$

where A_k^α, B_k^α, W_α, ω_α, λ_α is the set of $2N^2 + 3N$ unknown functions, denoted below by $Y(t)$. Substituting (1.2) into (1.1), we obtain the $2N^2 + 2N$ equations

$$\sum_\alpha W_\alpha\left[A_k^\alpha + \partial_\alpha\frac{B_k^\alpha}{\omega_\alpha}\right] = \delta_{k1},\quad \sum_\alpha\frac{W_\alpha B_k^\alpha}{\omega_\alpha} = 0 \tag{1.3}$$

$$U_{kn}A_n^\alpha = \omega_\alpha^2A_k^\alpha + \Delta_k^\alpha(A, B),\quad U_{kn}B_n^\alpha = \omega_\alpha^2B_k^\alpha + \Delta_k^\alpha(B, -A) \tag{1.4}$$
$$\Delta_k^\alpha(A, B) = -S_{kn}^\alpha A_n^\alpha + \omega_\alpha\nabla_{kn}^\alpha B_n^\alpha$$

Here we introduce the operators

$$\partial_\alpha = d/dt - \gamma_\alpha,\quad \nabla_{kn}^\alpha = 2(\delta_{kn}\partial_\alpha + R_{kn})$$
$$S_{kn}^\alpha = \delta_{kn}[\omega_\alpha^{1/2}\partial_\alpha^2\omega_\alpha^{-1/2} + {}^1/_2 m_k^{\cdot\cdot}/m_k - ({}^1/_2 m_k^{\cdot}/m_k)^2] + 2R_{kn}\omega_\alpha^{1/2}(\partial_\alpha - {}^1/_2 m_k^{\cdot}/m_k)\omega_\alpha^{-1/2}$$

We choose a further N equations such that $Y(t)$ depends on time only via the parameters, i.e., $Y(t) = Y(C, C^{\cdot}, \ldots)$. For this, we note that, by (1.4),

$$E_\alpha^{\cdot} = 2\gamma_\alpha E_\alpha - 2(R_{kn}A_k{}^\alpha A_n{}^\alpha + \rho_\alpha) \tag{1.5}$$

$$E_\alpha = A_n{}^\alpha A_n{}^\alpha + B_n{}^\alpha B_n{}^\alpha + \xi_{nn}^\alpha, \quad \xi_{nn}^\alpha = \frac{1}{\omega_\alpha}\left(A_k{}^\alpha \frac{dB_n{}^\alpha}{dt} - B_n{}^\alpha \frac{dA_k{}^\alpha}{dt}\right)$$

$$\rho_\alpha = R_{kn}[B_k{}^\alpha B_n{}^\alpha + \xi_{kn}^\alpha + m_n^{\cdot}(A_k{}^\alpha B_n{}^\alpha - A_n{}^\alpha B_k{}^\alpha)/2m_n]$$

It is clear from (1.5) that the requirement for implicit time-dependence can only be satisfied in the entire class of admissible functions (including quasistationary functions) if $E_\alpha = \text{const} > 0$. The choice of E_α in (1.2) is unimportant. Putting $E_\alpha = 1$, we obtain the N equations

$$\gamma_\alpha = R_{kn}A_k{}^\alpha A_n{}^\alpha + \rho_\alpha \tag{1.6}$$

We define the slowness of variation of the parameters and the smallness of the dissipative terms by introducing the small parameters ε_1, ε_2: $C \to C(\varepsilon_1 t)$, $R \to \varepsilon_2 R$, $0 \leqslant \varepsilon_i < \varepsilon_0$, $i = 1, 2$. Using algebraic transformations, we can write (1.3), (1.4), (1.6) as

$$\omega_\alpha = \sqrt{\lambda_\alpha} + \delta(\omega_\alpha), \quad \delta(\omega_\alpha) = -\Psi_k{}^\alpha \Delta_k{}^\alpha(A, B)/((\omega_\alpha + \sqrt{\lambda_\alpha})\Psi_n{}^\alpha A_n{}^\alpha) \tag{1.7}$$

$$\gamma_\alpha = \gamma_\alpha{}^\circ + \delta(\gamma_\alpha), \quad \gamma_\alpha{}^\circ = R_{kn}\Psi_k{}^\alpha \Psi_n{}^\alpha \tag{1.8}$$

$$\delta(\gamma_\alpha) = \rho_\alpha + R_{nk}[2\Psi_n{}^\alpha a_k{}^\alpha \sqrt{1-\varkappa_\alpha} - \Psi_n{}^\alpha \Psi_k{}^\alpha \varkappa_\alpha + a_n{}^\alpha a_k{}^\alpha]$$

$$A_k{}^\alpha = \Psi_k{}^\alpha + \delta(A_k{}^\alpha), \quad \delta(A_k{}^\alpha) = a_k{}^\alpha + \Psi_k{}^\alpha(\sqrt{1-\varkappa_\alpha} - 1) \tag{1.9}$$

$$W_\alpha = \Psi_\alpha + \delta(W_\alpha), \quad \delta(W_\alpha) = \frac{1}{\sqrt{1-\varkappa_\alpha}}\left(\frac{\varkappa_\alpha \Psi_\alpha}{1+\sqrt{1-\varkappa_\alpha}} - \sigma_k \Psi_k{}^\alpha\right) \tag{1.10}$$

$$B_k{}^\alpha = \sqrt{\lambda_\alpha}\left[\frac{\Psi_k{}^\alpha}{\Psi_\alpha}\sum_\beta \Psi_\beta g_n^{\alpha\beta}\Psi_n{}^\beta - \right. \tag{1.11}$$

$$\left. 2G_{kn}^\alpha\left(\left(\frac{d}{dt} - \gamma_\alpha{}^\circ\right)\Psi_n{}^\alpha + R_{nj}\Psi_j{}^\alpha\right)\right] + \delta(B_k{}^\alpha)$$

$$\delta(B_k{}^\alpha) = \frac{\Psi_k{}^\alpha}{W_\alpha}\sum_\beta \left\{\omega_\alpha W_\beta g_n^{\alpha\beta}\delta(A_n{}^\beta) - \frac{\omega_\alpha}{\omega_\beta}\Psi_n{}^\alpha b_n{}^\beta W_\beta + \right.$$

$$\left.\left[\sqrt{\lambda_\alpha}\left(\delta(W_\beta) - \frac{\Psi_\beta}{\Psi_\alpha}\delta(W_\alpha)\right) + W_\beta\delta(\omega_\alpha)\right]g_n^{\alpha\beta}A_n{}^\beta\right\} +$$

$$b_k{}^\alpha - G_{kn}^\alpha[\delta(\omega_\alpha)\nabla_{nj}^\alpha A_j{}^\alpha + \omega_\alpha \nabla_{nj}^\alpha \delta(A_j{}^\alpha) - 2\delta(\gamma_\alpha)\Psi_n{}^\alpha]$$

$$a_k{}^\alpha = G_{kn}^\alpha[(\omega_\alpha{}^2 - \lambda_\alpha)A_n{}^\alpha + \Delta_n{}^\alpha(A, B)], \quad G_{kn}^\alpha = P_{kn}^\alpha / P_1{}^\alpha$$

$$b_k{}^\alpha = G_{kn}^\alpha[(\omega_\alpha{}^2 - \lambda_\alpha)B_n{}^\alpha - S_{nj}^\alpha B_j{}^\alpha], \quad \sigma_k = \sum_\alpha W_\alpha\left(\partial_\alpha \frac{B_k{}^\alpha}{\omega_\alpha} - a_k{}^\alpha\right)$$

$$\varkappa_\alpha = B_n{}^\alpha B_n{}^\alpha + \xi_{nn}^\alpha + a_n{}^\alpha(2\Psi_n{}^\alpha A_1{}^\alpha/\Psi_\alpha + a_n{}^\alpha)$$

where the operator $g_n{}^{\alpha\beta} = \Psi_k{}^\alpha G_k{}^\alpha{}_j \nabla_{jn}{}^\beta$. Note that $a_1{}^\alpha = b_1{}^\alpha = 0$. We can prove the order relations: $(A, W, \omega) \sim 1$, $(B, \gamma) \sim \varepsilon$; $(\delta(A), \delta(W), \delta(\omega)) \sim \varepsilon^2$, $(\delta(B), \delta(\gamma)) \sim \varepsilon^3$, $\varepsilon = \varepsilon_1, \varepsilon_2$.

In short, in Eqs. (1.7)-(1.11), written in the form $Y = Y_1 + \delta(Y)$, the second terms are two orders smaller than the first. Using the procedure of successive approximation, we have

$$Y_{(1)} = Y_1, \ldots, Y_{(k+1)} = Y_1 + \delta(Y)|_{Y=Y_{(k)}}, \quad Y_1(t) = Y_1(C, C^{\cdot}) \tag{1.12}$$

$$|Y_{(k+1)} - Y_{(k)}| < \varepsilon^{2k+\delta}M_k, \quad k = 1, 2, \ldots, \quad \varepsilon = \max(\varepsilon_1, \varepsilon_2) \tag{1.13}$$

where $\delta = 0$ for $Y = A, W, \omega$ and $\delta = 1$ for $Y = B, \gamma$, while the M_k are bounded constants for all $0 \leqslant \varepsilon_i < \varepsilon_0$. Note that ε_0 and M_k depend on the properties of the functions $C(t)$ for $0 \leqslant t \leqslant T$.

It was assumed above that $\Psi_\alpha > 0$ ($|P_1{}^\alpha| \geqslant P_0 > 0$). The results are applicable, however, in the cases when $\Psi_\alpha \equiv 0$ for certain α (e.g., U is the direct sum of square matrices of lower order). Let $\Psi_1 \equiv 0$. We must then assume in (1.5) that $E_1 = 0$. To allow for this, it suffices, with $\alpha = 1$, to multiply Eqs. (1.9), (1.11) by $\Psi_1 = \text{const}$, divide (1.10) by the same, and to substitute the limits $(A_k{}^1\Psi_1, B_k{}^1\Psi_1, W_1/\Psi_1)|_{\Psi_1 \to 0}$ into (1.2), (1.7), (1.8) instead of $A_k{}^1$, $B_k{}^1$ and W_1 (the bounded limit $\Psi_1 G_{nk}{}^1|_{\Psi_1 \to 0}$ exists).

The problem is more difficult if the matrix U has multiple eigenvalues. Let $\lambda_1 = \lambda_2 =$

$\ldots = \lambda_s \geqslant \lambda_0 > 0,\ |\lambda_\alpha - \lambda_\beta| \geqslant \lambda_0$ for all $\alpha = 1, \ldots, N,\ \alpha \neq \beta \geqslant (s \pm 1)$. In this case, the functions $A_k^\alpha, B_k^\alpha, W_\alpha, \omega_\alpha, \lambda_\alpha$ with $\alpha \geqslant s+1$ may be found by the above method, while with $\alpha \leqslant s$ they are connected by a system of s differential equations. We take mutually orthogonal vectors Ψ_k^α such that $\Psi_n^\alpha R_{nk} \Psi_k^\alpha = r_\alpha \delta_{\alpha\beta}$, $\alpha, \beta = 1, 2, \ldots, s$. Then, putting

$$\gamma_\alpha = \gamma = s^{-1}(r_1 + \ldots + r_s), \quad \omega_\alpha = \sqrt{\lambda_1}, \quad B_k^\alpha = 0, \quad \alpha \leqslant s$$

we obtain the first approximation

$$A_k^\alpha = \sum_{\beta=1}^{s} \Psi_k^\beta H_{\beta\alpha}, \quad W_\alpha = \sum_{\beta=1}^{s} H_{\alpha\beta}^{-1} \Psi_\beta$$
$$H_{\alpha\beta}^{\cdot}(t) = (\gamma - r_\alpha) H_{\alpha\beta} + \sum_{\sigma=1}^{s} V_{\alpha\sigma} H_{\sigma\beta}, \quad \alpha, \beta = 1, \ldots, s \tag{1.14}$$
$$V_{\alpha\beta} = -V_{\beta\alpha} = \Psi_k^\beta \frac{d}{dt} \Psi_k^\alpha, \quad \det H(0) = 1$$

With $s = 2$ we obtain from (1.14)

$$H = \begin{Vmatrix} a_1 c_1 & -a_2 s_2 \\ a_1 s_1 & a_2 c_2 \end{Vmatrix}, \quad H^{-1} = \begin{Vmatrix} a_2 c_2 & a_2 s_2 \\ -a_1 s_1 & a_1 c_1 \end{Vmatrix}$$
$$c_\alpha = \cos\theta_\alpha, \quad s_\alpha = \sin\theta_\alpha$$
$$a_\alpha = \exp\left\{(-1)^\alpha \int_0^t b \cos(2\theta_\alpha)\, dt\right\}, \quad b = \frac{r_1 - r_2}{2}, \quad \alpha = 1, 2$$

where $\theta_\alpha(t)$ is found from the equations

$$\theta_\alpha^{\cdot} + (-1)^\alpha b \sin(2\theta_\alpha) = V_{21}, \quad \theta_\alpha(0) = 0, \quad \alpha = 1, 2$$

In (1.2) we put $F_k = \delta_{k1} F_1$. In the general case we have

$$x_k(t) = \int_0^t D_{kj}(t, t') F_j(t')\, dt'$$

The first column D_{k1} of the matrix D_{kj} is given in (1.2). The remaining columns are similar, except that throughout the subscript 1 has to be replaced by $j = 2, 3, \ldots, N$. In the eigenvectors, only the sign can vary, while $W_\alpha \simeq \Psi_\alpha \to \Psi_j^\alpha$.

To obtain the solution of Eq.(1.1) under arbitrary (bounded) initial conditions, it suffices to introduce into the sums over α in (1.2) the factors $\theta_\alpha = \text{const}$, and replace the zero lower limit in the integrals by $t_\alpha = \text{const}$. This is equivalent to adding to (1.2) the general solution of Eq.(1.1) with $F_k = 0$, which can be written as

$$\sum_{\alpha=1}^{N} \frac{K_\alpha e^{-\nu_\alpha}}{(m_k \omega_\alpha)^{1/2}} [A_k^\alpha \sin(\xi_\alpha(t) + \xi_{\alpha 0}) + B_k^\alpha \cos(\xi_\alpha(t) + \xi_{\alpha 0})] \tag{1.15}$$
$$\xi_\alpha = \int_0^t \omega_\alpha dt, \quad \nu_\alpha = \int_0^t \gamma_\alpha dt, \quad K_\alpha, \xi_{\alpha 0} = \text{const}$$

With $N = 1$, we obtain from (1.7)-(1.11), omitting the subscripts, $B = 0, A = W \equiv 1$, $\gamma = R$, and the general solution is written as

$$x = L^{-1} F_1 + \frac{K e^{-\nu}}{(m\omega)^{1/2}} \sin(\xi(t) + \xi_0) \tag{1.16}$$
$$L^{-1} F = \frac{e^{-\nu}}{(m\omega)^{1/2}} \int_0^t \frac{e^{\nu(t')} F(t')}{[m(t')\,\omega(t')]^{1/2}} \sin(\xi(t) - \xi(t'))\, dt' \tag{1.17}$$

where $\omega(t)$ is found by successive approximation from the equation

$$\omega^2 = b + \delta(\omega), \quad b = U - \gamma^2 - \frac{1}{m} \frac{d}{dt} \gamma m - \frac{1}{\sqrt{m}} \frac{d^2}{dt^2} \sqrt{m} \tag{1.18}$$
$$\delta(\omega) = \sqrt{\omega} \frac{d^2}{dt^2} \frac{1}{\sqrt{\omega}} \sim \varepsilon_1^2$$

In /2/ the resonance solutions are complex, while the expansion is with respect to ε (in (1.7)-(1.11) the expansion is with respect to ε^2). A detailed comparison is therefore difficult. But even in the first approximation, it can be seen that the results are different. For instance, with $F = e^{ipt}, p > 0, N = 1$, the particular solution in /1, 2/ in the form (1.2) is different, in what $W_1 = 2\sqrt{U}/(p+\sqrt{U}) \not\equiv 1$. The difference is most clearly marked in the stationary limit $(\varepsilon_1 \to 0)$; the solution (1.16) converges to the exact solution, while in the particular solutions of /1, 2/ the extra factor $2\sqrt{U}/(p+\sqrt{U})$ appears.

Let us consider the accuracy and asymptotic convergence of our results. Let $X = (x_1, \ldots, x_N)$ be the exact solution of Eq.(1.1), and $X_{(n)}$ the n-th approximation (i.e., in (1.2), $Y = Y_{(n)}$). Recalling (1.13), the asymptotic convergence can be proved: if $T = T_0/\varepsilon_1$ and $C(\tau)(\tau = \varepsilon_1 t)$ are $2n$ times differentiable with respect to τ for $0 \leqslant \tau \leqslant T_0$, then, for every T_0, we can find constants M_n, ε_0 such that $|X - X_{(n)}| < \varepsilon^{2n-1} M_n$ for all $0 \leqslant \varepsilon < \varepsilon_0$ (here and throughout, the similar inequalities for $|X^{\cdot} - X_{(n)}^{\cdot}|$ are omitted).

If we narrow down the problem, we can consider the concrete accuracy when the time interval is not restricted and it is assumed in essence that there are two distinct parameters $\varepsilon_1, \varepsilon_2$.

We define the operator $L_{(n)}^{-1}$ by substituting into (1.17) $\omega = \omega_{(n)} = \sqrt{q_n}$, where q_n is given by the sequence

$$q_1 = b, \ldots, \quad q_{k+1} = b + \delta_k, \ldots, \quad \delta_k = q_k^{1/4} \frac{d^2}{dt^2} q_k^{-1/4}, \quad k = 1, \ldots, n \tag{1.18'}$$

and we assume that $b(\tau)$ is differentiable a sufficient number of times with respect to τ for all $\tau \geqslant 0$. Let $x_{(n)}$ denote the solution of (1.16) for $\omega = \omega_{(n)}$, i.e., $L_{(n)} x_{(n)} = F$, where $L_{(n)} = L - \delta_n + \delta_{n-1}$ $(\delta_0 = 0)$.

Let

$$\langle\gamma\rangle = \left(\frac{1}{T}\int_0^T \gamma\, dt\right)\Big|_{T\to\infty}$$

exist, and for all $t \geqslant 0$ let

$$\left|\int_0^t (\gamma - \langle\gamma\rangle)\, dt\right| \leqslant z_0 = \text{const}, \quad 0 < m_0 \leqslant m(t) < \infty, \quad \langle\gamma\rangle = \varepsilon_2\gamma_0 > 0 \tag{1.19}$$

Then, given any bounded function $a(t)(|a(t)| \leqslant a_0 < \infty)$ and any n such that $0 < \omega_{\min} \leqslant \omega_{(n)}$, we have

$$|L_{(n)}^{-1} a| \leqslant \beta a_0/\varepsilon_2\gamma_0, \quad \beta = e^{2z_0}(m_0 \cdot \omega_{\min})^{-1}, \quad t \geqslant 0 \tag{1.20}$$

(a similar inequality holds for $|dL_{(n)}^{-1}a/dt|$).

Put $\alpha_n = \delta_n - \delta_{n-1}$. If $\omega_{(k)} \geqslant \omega_{\min} > 0$ for $k \leqslant n$, then $\beta \max|\alpha_n| = \varepsilon_1^{2n} b_n$, where the b_n are negative for all $0 \leqslant \varepsilon_i < \varepsilon_0$, $i = 1, 2$.

Theorem 1. Let the function $F(t)$ be bounded for all $t \geqslant 0$, let Conditions (1.19) hold, and for at least one $n = 1, 2, \ldots$, let

$$\varepsilon_1^{2n} b_n/\varepsilon_2\gamma_0 \leqslant s_0 < 1, \quad \omega_{(k)} \geqslant \omega_{\min} > 0, \quad k = 1, \ldots, n \tag{1.21}$$

Then, the exact solution of (1.1) with $N = 1$ is bounded, and for all n that satisfy (1.21) we have

$$|x_{(n)} - x| < \varepsilon_1^{2n} b_n \max|x_{(n)}|/(\varepsilon_2\gamma_0 - \varepsilon_1^{2n} b_n) \tag{1.22}$$

if $x_{(n)}(0) = x(0)$, $x_{(n)}^{\cdot}(0) = x^{\cdot}(0)$.

Proof. By (1.20), $x_{(n)}$ is bounded. For x we have

$$x = x_{(n)} - L_{(n)}^{-1}\alpha_n x = \sum_{k=0}^{\infty} (-L_{(n)}^{-1}\alpha_n)^k x_{(n)}$$

whence, using (1.20) and (1.21), we see that $x(t)$ is bounded, i.e., we obtain (1.22).

The operator L for which (1.19) and (1.21) hold, will be called bounded, and when separating the undamped solutions in (1.17) we extend the lower limit of integration to $-\infty$. Notice that, with $F = \exp(ipt)$, $|p - \omega| \geqslant \Delta_0 \gg \varepsilon_2\gamma_0$, (1.17) can be integrated by parts, and we obtain the estimate

$$|L^{-1}e^{ipt}| = \left|\frac{e^{ipt}}{\omega^2 - p^2} + \varepsilon(\ldots)\right| < \frac{\text{const}}{\Delta_0} \tag{1.23}$$

Theorem 1 demonstrates the different effect of the parameters ε_1 and ε_2 on the accuracy of the approximate solutions. If $\varepsilon_2 \ll \varepsilon_1$, the first approximations may be meaningless (the error is comparable to $\max|x|$). Since the a_n contain derivatives up to order $2n$, we can, in general, except for b_n a factorial growth: $b_n \sim (2n)!$ for $n \gg 1$, i.e., $\min(\varepsilon_1^{2n} b_n) \sim e^{-1/\varepsilon_1}$ for $n \sim {}^1/_2\varepsilon_1$, and $\varepsilon_1^{2n} b_{2n} \gg 1$ for $n > e/2\varepsilon_1$ (it is assumed that $\omega_{\min} \sim 1$, $\gamma_0 \sim 1$, $\varepsilon_1 \ll 1$). In this case the best approximations are connected with the values $n \sim {}^1/_2\varepsilon_1$, while if $n > e/2\varepsilon_1$, the $x_{(n)}$ may have no meaning. Then, for $x(t)$ to be bounded, it suffices to require that $\varepsilon_2 e^{1/\varepsilon_1} > 1$ as $(\varepsilon_1, \varepsilon_2) \to 0$.

In the multidimensional case, given suitable constraints, a theorem similar to Theorem 1 holds. Here, in the inequalities of the type (1.21); (1.22) we have $0 < \varepsilon_2\gamma_0 = \min\langle\gamma_\alpha\rangle$ $(\alpha = 1, \ldots, N)$, while

$$\varepsilon_1^{2n} b_n \to \sum_{k=0}^{n} \varepsilon_1^{2k} \varepsilon_2^{2(n-k)} b_n^{(k)}$$

where b_n^k are bounded for all $0 \leqslant \varepsilon_i < \varepsilon_0$.

The proof is based on the fact that the multidimensional analogue of $L_{(n)}$ differs from the exact operator by the operator $\varepsilon^{2n}\left(K_{(n)}\frac{d}{dt} + U_{(n)}\right)$, where the matrices $K_{(n)}$, $U_{(n)}$ are bounded for all $0 \leqslant \varepsilon_i < \varepsilon_0$, $i = 1, 2$, if $C(\tau)$ $(\tau = \varepsilon_1 t \geqslant 0)$ are differentiable $2n$ times.

2. Spectral analysis. Let the parameters $C(t)$ vary with frequency $\Omega \ll \min\omega_\alpha$, and let $F_1 = \exp(ipt)$, $p > 0$. We introduce functions

$$\varphi_\alpha = \int_0^t (\omega_\alpha - \omega_{\alpha 0})\,dt, \quad \eta_\alpha = \int_0^t (\gamma_\alpha - \gamma_{\alpha 0})\,dt, \quad \omega_{\alpha 0} = \langle\omega_\alpha\rangle, \quad \gamma_{\alpha 0} = \langle\gamma_\alpha\rangle$$

$$y_k^\alpha = \frac{(B_k^\alpha + iA_k^\alpha)\,e^{-\eta_\alpha}}{(m_k\omega_\alpha)^{1/2}}, \quad z_\alpha = \frac{W_\alpha e^{\eta_\alpha}}{(m_1\omega_\alpha)^{1/2}}, \quad \langle\cdot\rangle = \frac{\Omega}{2\pi}\int_0^{2\pi/\Omega} (\cdot)\,dt$$

$$\Phi_n(a) = \langle a(t)\exp(-in\Omega t)\rangle, \quad S_n^\alpha(a) = \Phi_n(ae^{i\varphi_\alpha})$$

Putting $\min\langle\gamma_\alpha\rangle > 0$, we obtain from (1.2) the spectral resolution of the undamped oscillations

$$x_k = \sum_{n=-\infty}^{+\infty} \exp[i(p + n\Omega)t] \sum_\alpha (f_{k,n}^\alpha(p) + \bar{f}_{k,-n}^\alpha(-p)) \tag{2.1}$$

$$f_{k,n}^\alpha(p) = \sum_{m=-\infty}^{+\infty} \frac{S_{n+m}^\alpha(\bar{y}_k^\alpha)\,\bar{S}_m^\alpha(z_\alpha)}{2(\omega_{\alpha 0} + m\Omega - p + i\gamma_{\alpha 0})} \tag{2.2}$$

For $|\gamma_\alpha| \lesssim \Omega$, it can be assumed that $|\eta_\alpha| \ll 1$. The functions $\varphi_\alpha \sim 1/\Omega$, i.e., for sufficiently small Ω the coefficients S_n^α can be found by the stationary phase method /3/. Note also that

$$\bar{f}_{k,-n}^\alpha(-p) \simeq \frac{\Phi_n(y_k^\alpha z_\alpha)}{(\omega_{\alpha 0} + p)}, \quad \sum_{n=-\infty}^{+\infty} e^{in\Omega t}\bar{f}_{k,-n}^\alpha(-p) \simeq \frac{y_k^\alpha z_\alpha}{\omega_\alpha + p}$$

If $\gamma_{\alpha 0} \ll \Omega$ the number of significant terms in the sums (2.2) is less, and in particular,

$$f_{k,j}^\alpha(\omega_{\alpha n}) \simeq S_{j+n}^\alpha(\bar{y}_k^\alpha)\,\bar{S}_n^\alpha(z_\alpha)/(2i\gamma_{\alpha 0}), \quad \omega_{\alpha n} = \omega_{\alpha 0} + n\Omega$$

In the one-dimensional system with $m_1 = 1$ and $\gamma = \text{const} > 0$, we have

$$x = \sum_n \exp[i(p + n\Omega)t](f_n(p) - \bar{f}_{-n}(-p)), \quad f_n(p) = \sum_k \frac{S_{n+k}\bar{S}_k}{2h_k(p)} \tag{2.3}$$

$$h_k(p) = \omega_k + i\gamma - p, \quad \omega_k = \langle\omega\rangle + k\Omega$$

$$S_n = \Phi_n\left(\omega^{-1/2}\exp\left[\int_0^t (\omega - \omega_0)\,dt\right]\right)$$

The quantities $\mu^{(+)} = (\max\omega - \omega_0)/\Omega$, $\mu^{(-)} = (\omega_0 - \min\omega)/\Omega$ characterize the number of significant harmonics in (2.3): $S_n \to 0$ for $n > \mu^{(+)}$ and $n < -\mu^{(-)}$. It can be assumed without loss of generality that $\mu^{(+)} \simeq \mu^{(-)} = \mu$ (μ is sometimes called the excitation level). If $|p - \omega_0| > \mu\Omega$ or $\Omega \ll \gamma$, we obtain from (2.3) the stationary approximation $x = (\omega^2 + (\gamma + ip)^2)^{-1}e^{ipt}$. Thus, analysis of the resonant domain $|p - \omega_0| \lesssim \mu\Omega$ with $\gamma \lesssim \Omega$ is more important. If $\gamma \ll \omega_0/\mu$, we have for the function $E(p) = \langle |x|^2\rangle$ (in the stationary case this is the amplitude-frequency response) the relation

$$E = \frac{1}{4}\sum_k\sum_n \frac{\Phi_{n-k}(\omega^{-1})S_n\bar{S}_k}{h_k\bar{h}_n} \simeq \frac{\Phi_0}{4}\sum_n\left|\frac{S_n}{h_n}\right|^2, \quad \Phi_0 = \left\langle\frac{1}{\omega}\right\rangle$$

It can be seen that the maxima of $E(p)$ are linked with the values $p = \omega_n = \langle\omega\rangle + n\Omega$. The frequencies ω_n ($|n| \leqslant \mu$) may be called resonant. Not all the resonances appear: if $S_k \simeq 0$ for some k ($|k| \lesssim \mu$), we obtain instead of a maximum for $p = \omega_k$ an extra deeper minimum. When $\mu \gg 1$, the curve $E(p)$ is nearer the boundary of the resonant domain at the top than at the middle of the domain. If $\Omega \gg \gamma$, there is a notable similarity between the spectral amplitudes and the maxima at the resonant frequencies. In fact

$$E_{(n)} = E(\omega_n) \simeq \Phi_0|S_n|^2/(4\gamma^2), \quad f_n{}^k = f_{n-k}(\omega_k) \simeq \frac{S_nS_k}{2i\gamma}$$

$$|f_n{}^k|^2 \approx 4E_{(n)}F_{(k)}\gamma^2/\Phi_0 \to |f_n{}^k/f_n{}^j|^2 = |f_k{}^n/f_j{}^n|^2 = \frac{E_{(k)}}{E_{(j)}}$$

Here, $f_n{}^k$ is the complex amplitude of the harmonic $\exp(i\omega_n t)$ for $p = \omega_k$ ($|k| \lesssim \mu$), so that, by calculating (or measuring experimentally) the amplitudes $|f_n{}^k|$ for some $|k| \leqslant \mu$, we can also estimate $|f_n{}^j|$ for $j \neq k$, and also the behaviour of the curve $E(p)$ (the size of the maxima, and the position of the extra minima, etc.).

The forced oscillations in multidimensional systems are made up of one-dimensional oscillations, to which correspond the natural frequencies ω_α and the coefficients of friction γ_α. If all the parameters vary over a small range with the same frequency Ω, the quantities

$$\langle\gamma_\alpha\rangle, \quad \omega_{\alpha n} = \langle\omega_\alpha\rangle + n\Omega, \quad S_n{}^\alpha(\omega_\alpha^{-1/2}), \quad \mu_\alpha = \max|\omega_\alpha - \omega_{\alpha 0}|/\Omega$$

define the main properties of the spectral amplitudes of these oscillations. Every mean square characteristic $E(p) = g_{kn}\langle x_k\bar{x}_n\rangle$ ($g_{kn} = \bar{g}_{nk} = \text{const}$) in the ranges of p where $|\Omega E_0^{-1}dE_0/dp| \leqslant \varkappa \ll 1$ ($E_0(p)$ is the stationary analogue of $E(p)$) is virtually equal to $E_0(p)$ ($|E - E_0| \lesssim \varkappa E_0$). In the resonance domains ($|p - \omega_{\alpha 0}| \lesssim \mu_\alpha\Omega$) however, $E(p)$ is qualitatively different from $E_0(p)$, and we have

$$E(p) \simeq \frac{1}{\langle\omega_\alpha^{-1}\rangle}\sum|S_n{}^\alpha|^2E_0(p + n\Omega)$$

if $\mu_\alpha \gg 1$, $|p - \omega_{\alpha 0}| \leqslant \mu_\alpha\Omega$, $|\omega_{\alpha 0} - \omega_{\beta 0}| > (\mu_\alpha + \mu_\beta)\Omega$, $\beta \neq \alpha$.

The above spectral analysis gives an idea of how the properties of the forced oscillations depend on the functions $Y = (A, B, W, \omega, \lambda)$. The most important characteristics are the functions

$$\varphi_\alpha(t) = \int_0^t(\omega_\alpha - \langle\omega\rangle)\,dt$$

(phase oscillations) and the means $\langle Y\rangle$.

3. *Non-linear disturbance*. We shall study the non-linear oscillations by using the above method of partial linearization. We shall confine ourselves to the one-dimensional case; under suitable conditions, our scheme can be easily extended to the cases $N \geqslant 2$.

The non-linear generalization of Eq.(1.1) with $N = m_1 \doteq 1$ is

$$Lx = \left(\frac{d^2}{dt^2} + 2\varepsilon\gamma(\tau)\frac{d}{dt} + U(\tau)\right)x = F(t) + \varepsilon Q(x, x^{\cdot}, t), \quad \tau = \varepsilon t \tag{3.1}$$

where $Q(x, x^{\cdot}, t)$ is infinitely differentiable with respect to x and $x^{\cdot}$ (in particular, it may be polynomial), and $Q(0, 0, t) = 0$. We note at one that, if x_1 is a bounded solution of Eq.(3.1), then in general $x = x_1 + y$, where $y(t)$ is given by

$$Ly = \varepsilon V(x_1, y),\ V(x_1, y) = Q(x_1 + y,\ x_1^{\cdot} + y^{\cdot},\ t) - Q(x_1, x_1^{\cdot}, t) \tag{3.2}$$

We will first introduce some notation. Let $\omega(t)$ be an approximation, given in (1.18). We assume that $0 < \varepsilon\gamma_0 = \langle\gamma\rangle\varepsilon \leqslant \omega_{\min} \leqslant \omega \leqslant \omega_{\max}$. Let g be the frequency interval $(\omega_{\min} - \Delta_0, \omega_{\max} + \Delta_0)$, where $\Delta_0 = \text{const}$ satisfies the conditions $\varepsilon\gamma_0 \ll \Delta_0 \ll \omega_{\min}$. Let G be the set of infinitely differentiable functions which can be expanded for $t \geqslant 0$ in absolutely convergent series of the type $\sum A_k \cos(\theta_k t + \varphi_k)$. If $|\theta_k| \in g$ in these series, then $G_1 \subset G$, while if $|\theta_k| < \Delta_0$, then $G_0 \subset G$. For a function $g(t) \in G$ we define the operations Hq, $\{q\}$: $Hq \in G_1$, $\{q\} \in G - G_1$, $q = Hq + \{q\}$ (i.e., Hq and $\{q\}$ are the resonant and non-resonant parts of $q(t)$). We also introduce $H'q$ and $\{q\}' = (1 - H')q$ such that $LH'q \in G_1$, $L\{q\}' \in G - G_1$.

A "smooth" division of the frequency spectrum is often more convenient: we preserve in Hq and $\{q\}$ the harmonics of rapidly decreasing amplitude, whose frequencies go beyond the indicated limits. In such cases we are usually dealing with functions of the type $\sum B_k \cos(p_k t + \psi_k)$, where $(B_k, \psi_k) \in G_0$ (the class of slowly varying functions G_0 can also be smoothed). We can then define operators H_n, which leave unchanged the harmonics with frequencies close to $n\langle\omega\rangle$: $H_n q = B\cos(n\langle\omega\rangle t + \psi)$, $(B, \psi) \in G_0$; sometimes, $H' = L^{-1}H \simeq H_1$.

In (3.1) let $(\gamma, U) \in G_0$, $(F(t), Q(t)) \in G$ for all $t \geqslant 0$. We introduce the idea of frequency linearization as follows: if $Ly \in G_1$ and $z \in G$, then

$$HV(z, y) = uy + 2ry^{\cdot},\ (u, r) \in G_0,\ u = u(z, y),\ r = r(z, y) \tag{3.3}$$

In general,

$$\begin{gathered}
HV = \sum A_k \cos(\xi_k t + \psi_k),\quad y = \sum_k Y_k \cos(\Theta_k t + \varphi_k),\\
u = \frac{1}{\rho}\sum_k\sum_m Y_k A_m \Theta_k \cos[(\xi_m - \Theta_k)t + \psi_m - \varphi_k]\\
r = \frac{1}{2\rho}\sum_k\sum_m Y_k A_m \sin[(\xi_m - \Theta_k)t + \psi_m - \varphi_k]\\
\rho = \sum_k\sum_m Y_k Y_m \Theta_k \cos[(\Theta_m - \Theta_k)t + \varphi_m - \varphi_k],\quad (\xi_k, \Theta_k) \in g
\end{gathered} \tag{3.4}$$

If "smooth" separation is possible and $y = a\cos(pt + \varphi)$, $p \simeq \langle\omega\rangle$, $(a, \varphi) \in G_0$, $a \geqslant a_0 > 0$ (we then have in (3.4) $\rho = a(p + \varphi^{\cdot})$), then, putting

$$\begin{gathered}
V_1 = \frac{1}{2y}(V + B),\quad V_2 = \frac{1}{2y^{\cdot}}(V - B),\quad B = Q(z + y, z^{\cdot}, t) - Q(z, z^{\cdot} + y^{\cdot}, t)\\
(H_0 + H_2)V_i = a_i + b_i\cos(2pt + 2\varphi + \varphi_i),\quad (a_i, b_i, \varphi_i) \in G_0,\quad i = 1, 2
\end{gathered}$$

and observing that $V_1 \to \partial Q(z, z^{\cdot}, t)/\partial z$, $V_2 \to \partial Q(z, z^{\cdot}, t)/\partial z^{\cdot}$ as $y \to 0$, we have

$$\begin{gathered}
r = \frac{a_2}{2} - \frac{b_2}{4}\cos\varphi_2 + \frac{1}{4(p + \varphi^{\cdot})}\left(b_1\sin\varphi_1 + b_2\frac{a^{\cdot}}{a}\sin\varphi_2\right)\\
u = a_1 + a_2\frac{a^{\cdot}}{a} + \frac{b_1}{2}\cos\varphi_1 + \frac{b_2}{2}\left(\frac{a^{\cdot}}{a}\cos\varphi_2 + (p + \varphi^{\cdot})\sin\varphi_2\right) - \frac{2a^{\cdot}}{a}r
\end{gathered} \tag{3.5}$$

Now, putting $F = \varepsilon F_1 + F_2$, $\varepsilon F_1 = HF$, $F_2 = \{F\}$, the following scheme for analysing (3.1) can be proposed:

$$\begin{gathered}
x_0 = 0,\quad x_n = y_n + z_n,\quad y_n = H'x_n,\quad n = 1, 2, \ldots\\
z_n = L^{-1}(F_2 + \varepsilon\{Q(x_{n-1}, x_{n-1}^{\cdot}, t)\});\quad y_n = \varepsilon L_n^{-1}P_n\\
P_n = F_1 + HQ(z_n, z_n^{\cdot}, t),\quad L_n = L - 2\varepsilon r_n\frac{d}{dt} - \varepsilon u_n\\
r_n = r(z_n, y_n),\quad u_n = u(z_n, y_n)
\end{gathered} \tag{3.6}$$

When proving this procedure (see below), inequalities of the type $|q| < \text{const}$ simultaneously imply $(|q| + |q^{\cdot}|) < \text{const}$, if $q(t) \in G$.

Put

$$L(z, y) = L - 2\varepsilon r(z, y)\frac{d}{dt} - \varepsilon u(z, y), \quad L(z) = L(z, 0),$$

The operators $L, L_n, \ldots,$ will be said to be bounded, and we shall write $L < I$, $L_n < I$, ..., if, given any $g(t) \in G$, there are constants ε_0, K_1, K_2 such that

$$|L^{-1}\{q\}| < K_1 \max|q|, \ \varepsilon|L^{-1}Hq| < K_2 \max|q|, \ldots, 0 \leqslant \varepsilon < \varepsilon_0 \tag{3.7}$$

In particular, recalling (1.20) and (1.23), we see that (1.19) and (1.21) can serve as criteria for boundedness.

In (3.6), $(u_n, r_n) = C_n = C(z_n, y_n)$, $y_n = y(C_n, z_n, P_n)$. In the general case, these functional equations lead to a set of solutions, i.e., $C_n \to C_n{}^i = \Phi^{(i)}(z_n, P_n)$, where $\Phi^{(1)}$, $\Phi^{(2)}$, ..., is a set of functionals, the number of which depends on the type of non-linearity Henceforth, we shall understand by x_n any of the relevant sequences $x_n{}^{(1)}, x_n{}^{(2)}, \ldots$

Before starting our main theorem, we consider the equation

$$Ly = \varepsilon HV(z, y) + \varepsilon^{k+1}q(t) \to L(z, y)y = \varepsilon^{k+1}q, \quad z \in G, \ q \in G_1 \ (k = 1, 2, \ldots) \tag{3.8}$$

We shall write $L(z) < I^k$ $(k = 1, 2, \ldots)$ if, given any $\varphi(t) \in G_1$, there exists $\varepsilon_0 > 0$ such that $L(z, \varepsilon^k\varphi) < I$ for $\varepsilon < \varepsilon_0$ (i.e., the operator $L(z)$ is bounded with a "margin"). In the case of (3.8), this means that, among the functionals $\Phi^{(i)}$ there is one $\Phi^{(1)}(z, P)$, continuous in the neighbourhood $|P| < \varepsilon_0{}^k \max|q|$, such that $|\Phi^{(1)}(z, \varepsilon^k q)| < \varepsilon^k$ const. Hence we have

Lemma. If $L(z) < I^k$, then among the bounded solutions of (3.8) there is $y(t)$ such that $|y| < \varepsilon^k M_0$, where $M_0 = \text{const}$, $0 \leqslant \varepsilon < \varepsilon_0$.

We can now prove the asymptotic convergence of the procedure (3.6).

Theorem 2. If $(F_1, F_2) \in G$ for $0 \leqslant \varepsilon < \varepsilon_0$, then, for all the $0 \leqslant \varepsilon < \varepsilon_0$ for which $L_k < I$, $L(x_k) < I^k$ $(k = 1, 2, \ldots, n)$, we can find constants $M_n < \infty$ such that, among the bounded solutions of (3.1) there exists $x(t)$ such that $|\delta x_n| = |x - x_n| < \varepsilon^n M_n$.

Proof. For $\delta z_k = \{x\}' - z_k$, $\delta y_k = H'x - y_k$ we have

$$L\delta z_1 = \varepsilon\{Q(x, x^{\cdot}, t)\}, \ldots, L\delta z_{k+1} = \varepsilon\{V(x_k, \delta x_k)\} \tag{3.9}$$

$$L\delta y_k = \varepsilon HV(x_k, \delta y_k) - \varepsilon HV(x, -\delta z_k), \quad k = 1, \ldots, n \tag{3.10}$$

Since $Q(z, z^{\cdot}, t)$ is differentiable, we have the Lipschitz condition $|V(a, b)| < |b|$ const. Hence, successively analysing (3.9) and (3.10) ((3.9) for $\delta z_1 \to (3.10)$, for $\delta y_1 \to (3.9)$, for $\delta z_2 \to$ etc.), we obtain from (3.9) in the light of (3.7), $|\delta z_k| < \varepsilon^k M_k'$, and from (3.10) we have by the lemma, $|\delta y_k| < \varepsilon^k M_k''$ (M_k', M_k'' are constants). As a result,

$$|\delta x_k| < M_k\varepsilon^k, \quad k = 1, \ldots, n, \quad M_k = \text{const}$$

A few words about Eq.(3.2), where we take $L<I$ and seek $y \in G$. If $y = \{y\}' = z$, then $|z| \leqslant \varepsilon|z|$ const and $z = 0$ for sufficiently small ε. Hence $H'y = y_1 \not\equiv 0$ if $y \not\equiv 0$. Here, $L(x, y_1)y_1 = \varepsilon^2 q(t)$, $q \in G_1$ and (3.2) has non-trivial bounded solutions only when $L(x, y_1) < I$ as $\varepsilon \to 0$. For (3.2) we can propose a scheme similar to (3.6), and obtain in the first approximation $L(x, y_1)y_1 = 0$. Hence the wanted solutions clearly arise in the parameter domains which separate the cases of increasing and damped solutions of the equation $L(x, y_1)y_1 = 0$. These domains can be defined by the relation $(\langle\gamma - r(x, y_1)\rangle) \to +0$, $\varepsilon \to 0$, which will obviously not hold for any types of functions $x(t)$, $Q(x, x^{\cdot}, t)$.

For instance, we can put $F \to F + \alpha\varphi(t)$ in (3.1) and choose $\varphi(t) \in G$ in such a way that the relation is not satisfied. Passing to the limit as $\alpha \to 0$ in the results of (3.6), an increase in the number of functionals $\Phi^{(i)}$ must, in general, be expected. This procedure allows the scheme (3.6) to be used for seeking the solutions (3.2).

Functions of the class G have been considered above, so that the time interval has not been restricted. If we take $0 \leqslant t \leqslant T_0/\varepsilon$, $\tau = \varepsilon t$, all our results apply for functions $q(t, \tau) = q_\tau(t) \in G$, which are infinitely differentiable with respect to τ for all $0 \leqslant \tau \leqslant T_0$.

Now consider in more detail the one-dimensional system with

$$0 < \gamma = \text{const}, \quad \omega = 1 + \sum_{n \neq 0} in\Omega\mu_n \exp(in\Omega t), \quad \mu_n = \bar{\mu}_{-n} = \text{const}$$

$$\min \omega(t) = \omega_{\min} > 0, \quad 0 < \Omega \leqslant \omega_{\min}$$

$$Q = x^3, \quad F = \varepsilon F_0 \cos pt, \quad p \in g$$

In (3.6) the first approximation gives

$$z_1 = 0,\ r_1 = 0,\ x_1 = \varepsilon F_0 \operatorname{Re} \sum_n f_n \exp(i(p + n\Omega)t),\ f_n = \sum_k \frac{S_{n+k}\bar{S}_k}{2h_k(p + \varkappa_0)} \tag{3.11}$$

$$S_n = \Phi_n(\omega^{-1/2} \exp(i(\varphi - \varphi_1))),\ h_k(a) = 1 + k\Omega + i\varepsilon\gamma - a$$

$$\varphi = \int_0^t (\omega - 1)\, dt, \quad \varphi_1 = \int_0^t \left(\frac{\varepsilon u_1}{2\omega} - \varkappa_0\right) dt, \quad \varkappa_0 = \frac{\varepsilon}{2}\left\langle \frac{u_1}{\omega} \right\rangle$$

$$u_1 = \frac{3\varepsilon^2 F_0^2}{16\omega} \exp(-2\varepsilon\gamma t) \left| \int_{-\infty}^t \exp(\varepsilon\gamma t + i\psi)\, dt \right|^2 = \frac{3}{2} \sum_n E_n \exp(in\Omega t) \tag{3.12}$$

$$\psi = \varphi - \varphi_1 + (\omega_0 - p - \varkappa_0)t, \quad E_n = \frac{\varepsilon^2 F_0^2}{2} \sum_k f_{n+k}\bar{f}_k = \langle x^2 \exp(-in\Omega t)\rangle$$

Let the parameters vary over a small range: $|\omega - 1| \ll 1$ (in practice it suffices that $|\omega - 1| < 0.3$). Introducing for the set $\beta = (\beta_k)$, where $\beta_k = \beta_{-k} = \text{const}$, $k = \pm 1, \pm 2, \ldots,$ the notation

$$S_n(\beta) = \Phi_n\left(\exp\left[i \sum_{k \neq 0} \beta_k \exp(ik\Omega t)\right]\right), \quad f_n(a, \beta) = \sum_m \frac{S_{n+m}(\beta)\bar{S}_m(\beta)}{2h_m(a)}$$

$$F_n(a, \beta) = \frac{\varepsilon^2 F_0^2}{8} \sum_m \frac{S_{n+m}(\beta)\bar{S}_m(\beta)}{h_m(a)\bar{h}_{n+m}(a)},$$

we obtain $f_n = f_n(p + \varkappa_0, \mu - \varkappa)$ and the system of equations for $\varkappa_0$, $(\varkappa_k)$:

$$\varkappa_0 = \frac{3}{4} \varepsilon E_0(p + \varkappa_0, \mu - \varkappa), \quad \varkappa_k = -\frac{3\varepsilon i}{4k\Omega} E_k(p + \varkappa_0, \mu - \varkappa) \tag{3.13}$$

When $\Omega > \varepsilon\gamma$ we have the order relations $|f_n| \lesssim (2\varepsilon\gamma)^{-1}$, $|\varkappa_0| \lesssim 3\varepsilon F_0^2/(32\gamma^2) = \alpha$, $|\varkappa_k| \lesssim \varepsilon\gamma\alpha/(k\Omega)^2$, where we take $\alpha \ll 1$. Obviously, it is only when $\gamma \ll 3\varepsilon^2 F_0^2/(32\Omega^2)$ that there is any point in taking account of $\varkappa_k$ with $|k| \gg 1$. If $\alpha\gamma/\Omega^2 < 1$ and $\omega = 1 + \Omega\mu_1 \cos\Omega t$, $\mu_1 \gg 1$, then, assuming that $\varkappa_k = 0$ for $|k| \geqslant 2$, we obtain $S_n = \exp(-in\psi_0) J_n(\mu_1 - \varkappa_1)$, $\psi_0 = \dfrac{3\varepsilon}{2\mu_1\Omega} \operatorname{Im} E_1(p + \varkappa_0, \mu_1 - \varkappa_1)$ (J_n are Bessel functions), and the system of equations for $\varkappa_0$, $\varkappa_1$:

$$\varkappa_0 = \frac{3\varepsilon}{4} E_0(p + \varkappa_0, \mu_1 - \varkappa_1), \quad \varkappa_1 = \frac{3\varepsilon}{2\Omega} \operatorname{Re} E_1(p + \varkappa_0, \mu_1 - \varkappa_1) \tag{3.14}$$

Notice that (3.13) and (3.14) are in essence a parametric specification of the functions $\varkappa_n = \varkappa_n(p, \mu)$, $n = 0, \pm 1, \ldots$, $\mu = (\mu_k)$.

If the parameters can vary over a fairly wide range and $\max|\omega - 1| \gg \Omega$, the stationary phase method can be used to solve the functional Eq.(3.12). We shall assume that $\omega(t)$ is an even function, with one extremum in the half-period π/Ω. Assume that $\omega(t) = p$ gives $t = \pm t_p$, $0 \leqslant t_p < \pi/\Omega$, and that $w = \omega(t_p) \neq 0$. Then, if $|w| \gg \Omega^2$ (i.e., $\min\omega < p < \max\omega$), we obtain

$$u_1(t) = \frac{D\eta(t)}{\omega} e^{-2\varepsilon\gamma t}, \quad D = \frac{3\pi\varepsilon^2 F_0^2}{8p|w|} (\operatorname{ch} \varepsilon\gamma T - \cos yT)^{-1}$$

$$T = 2\pi/\Omega, \quad y = 1 - p - \varkappa_0$$

$$\eta(t) = \begin{cases} D_1 \exp(2\varepsilon\gamma Tn), & (nT - t_p) < t < (nT + t_p) \\ D_2 \exp(2\varepsilon\gamma Tn), & (nT + t_p) < t < (nT + T - t_p) \end{cases}$$

$$n_0 = 0, \pm 1, \ldots, D_1 = \operatorname{ch}(\varepsilon\gamma(2t_p - T)) + \cos(y(2t_p - T) + 2\psi - 2\beta)$$

$$D_2 = e^{\varepsilon\gamma T}[\operatorname{ch}(2\varepsilon\gamma t_p) + \cos(2yt_p + 2\psi - 2\beta)]$$

$$\psi = \int_0^{t_p} (\omega - 1)\, dt + \frac{\pi w}{4|w|}$$

Here, $\varkappa_0$, β are given by the system of equations

$$\varkappa_0 = \frac{\varepsilon}{2}(b_1 + b_2)D, \quad \beta = \frac{\varepsilon}{2}\left(\left(\frac{T}{2} - t_p\right)b_1 - t_p b_2\right)D$$

$$b_1 = \frac{D_1}{T} \int_{-t_p}^{t_p} q\, dt, \quad b_2 = \frac{D_2}{T} \int_{t_p}^{T - t_p} q\, dt, \quad q = \frac{\exp(-2\varepsilon\gamma t)}{\omega^2}$$

To sum up, we have demonstrated the cases when we can pass from the functional equations for $C_1 = (u_1, r_1)$ to a system of ordinary (not differential) equations with only a few unknowns. It can be said in general that the passage can be made if, in the expansion $C_1 = \Sigma A_k \cos(\theta_k t + \psi_k)$, the condition $|\varepsilon A_k| \geqslant \theta_k$ is satisfied by only a few harmonics. The stationary-phase method also simplifies the functional problem. Given these possibilities, our scheme is preferable to the methods described in /1/, in which the results are stated as first-order non-linear equations for the amplitudes and phases.

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THE CONDITION FOR SIGN-DEFINITENESS OF INTEGRAL QUADRATIC FORMS AND THE STABILITY OF DISTRIBUTED-PARAMETER SYSTEMS*

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The stability of distributed-parameter systems described by linear partial differential equations is investigated by reducing the original equations by a change of variables to a system of first-order equations in time and in spatial coordinates. The Lyapunov functions are constructed in the form of single integral forms. New necessary and sufficient conditions for the sign-definiteness of these forms are obtained. These conditions, unlike the Sylvester criterion, do not require the calculation of determinants. The check for sign-definiteness is made using recurrence relationships and is a generalization of the results obtained in /1/.

The proposed criteria are applied to derive sufficient conditions for the stability of distributed-parameter linear systems. The construction of functionals for the one-dimensional second-order linear hyperbolic equation is considered in more detail. As an example, we examine the stability of the torsional oscillations of an aircraft wing.

1. Consider a system of first-order linear partial differential equations of the form

$$\frac{\partial \varphi}{\partial t} = \sum_{k=1}^{s} \left(A_k(\mathbf{x}) \frac{\partial \varphi}{\partial x_k} + B_k(\mathbf{x}) \frac{\partial \psi}{\partial x_k} \right) + A_0(\mathbf{x}) \varphi + B_0(\mathbf{x}) \psi \tag{1.1}$$

$$\sum_{k=1}^{s} \left(C_k(\mathbf{x}) \frac{\partial \varphi}{\partial x_k} + D_k(\mathbf{x}) \frac{\partial \psi}{\partial x_k} \right) + C_0(\mathbf{x}) \varphi + D_0(\mathbf{x}) \psi = 0 \tag{1.2}$$

where $t \in I = (0, \infty)$, $\mathbf{x} = (x_1, x_2, \ldots, x_s)^T \in X \subset E^s$ is a vector of spatial coordinates, $\varphi = \varphi(\mathbf{x}, t)$ is the n-dimensional vector of phase functions, $\psi = \psi(\mathbf{x}, t)$ is the m-dimensional vector of phase functions whose derivative with respect to time does not occur in the system (1.1), (1.2), $A_k(\mathbf{x}), B_k(\mathbf{x}), C_k(\mathbf{x})$, and $D_k(\mathbf{x})$ $(k = 0, 1, \ldots, s)$ are matrices whose elements

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